INTEGRAL CUP

# Integral Cup 2025 – Phase 1 Preliminary Exam

## Question Paper & Answer Key

**Q1.** $a = \int_0^1 \frac{\arctan x}{x}\, dx$

$$\textit{Find} \sum_{r=0}^{\infty} \frac{1}{(4r+1)^2} \textit{ in terms of } a,$$

$$\textit{given that} \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$$

a. $0.616 + \frac{a}{2}$

b. $0.739 + \frac{a}{2}$

c. $0.232 + \frac{a}{2}$

d. $0.114 + \frac{a}{2}$

**Q2.** $\textit{Evaluate the integral}: \int_{-\infty}^{0} \frac{\sin^3 x}{x}\, dx$

Given $\pi = 3.14159265$

a. 0.785808

b. 0.785087

c. 0.785398

d. 0.785420



+91-9257282136

Info@vedantsamanvaya.com

**Q3.** $Evaluate\ the\ integral: \int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\ dx$

a. $\dfrac{\sqrt{\pi}\cdot e^{-\left(\frac{4ac-b^2}{4a}\right)}}{\sqrt{a}}$

b. $\dfrac{\sqrt{\pi}\cdot e^{-\left(\frac{b^2-4ac}{4a}\right)}}{\sqrt{a}}$

c. $\dfrac{\sqrt{\pi}\cdot e^{-\left(\frac{4ac-b^2}{2a}\right)}}{\sqrt{a}}$

d. $\dfrac{\sqrt{\pi}\cdot e^{-\left(\frac{4ac+b^2}{4a}\right)}}{\sqrt{a}}$

**Q4 .** $Evaluate\ the\ integral \int_{-\infty}^{0} \ln(1-e^{nx})\ dx$

$$given \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$$

a. $\frac{\pi^2}{6n}$

b. $-\frac{\pi^2}{6n}$

c. $-\ \frac{ln^2(n)}{2}$

d. $\frac{1}{n^2}$



+91-9257282136

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**Q5.** Evaluate the integral, given that the infinite sum of the sixth powers of the reciprocals of natural numbers is a.

$$\int_0^\infty \frac{x^5}{e^x - 1}\,dx$$

a. 120a

b. 48a

c. 240a

d. 216a

**Q6.** Evaluate the integral

$$\int_1^{\sqrt{\pi}} e^{x\sin(x^2)}\left(\sin(x^2) + 2x^2\cos(x^2)\right)dx$$

a. $e^{\sin(\sqrt{\pi})} - e^{\sin(1)}$

b. $e^{\sqrt{\pi}} - e^1$

c. $1 - e^{\sin(1)}$

d. $e^\pi - e$



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**Q7.** Find y(1)

$$\int y\,dx = xy - \int \frac{y^2}{1 - y\ln(x)}\,dx$$

Find y(1)

a. 1

b. 2

c. -1

d. 0

**Q8.** Assume left integral is A and right one is B then,

$$Let\ (\ f(x)\ )\ be\ a\ continuously\ differentiable\ function\ that\ is\ always\ positive.$$

$$Define\ H(x)\ =\ \sum_{i=0}^{\lfloor x \rfloor} f'(c_i)$$

$$where\ (\ c_i\ )\ are\ the\ constants\ obtained\ from\ the\ Lagrange\ Mean\ Value\ Theorem\ (LMVT)$$

$$in\ the\ intervals\ (\ [0,\ 1], [1,2], \ldots, [\lfloor x \rfloor, x]).$$

$$Compare\ the\ integrals:$$

$$\int_0^1 H(x)dx \quad \text{and} \quad \int_0^1 f(\lfloor x \rfloor)dx\,.$$

a. A = B

b. A > B

c. A < B

d. A <= B



+91-9257282136

Info@vedantsamanvaya.com

**Q9.** Compare the following two integrals

$$A = \int_{-\infty}^{\infty} e^{x(i-x)}\ dx \qquad B = \int_{-\infty}^{\infty} e^{-x^2} \cos x\ dx$$

a. A <= B

b. A > B

c. A is divergent

d. A is complex and B is real

**Q10.** Given the curve: $y = e^{-x^2}$

$$Let\ the\ point\ (A)\ be\ given\ as{:}\ A\left(\sqrt{\ln 2}, 0.5\right)$$

$$Consider\ a\ point\ (B\ )\ on\ the\ curve\ such\ that\ (\ y\ >\ 0.5\ ).\ From\ (\ A\ ),\ draw\ a$$

$$line\ parallel\ to\ the\ y\ -\ axis,\ and\ from\ (\ B\ ),\ draw\ a\ line\ parallel\ to\ the\ x$$
$$-\ axis\ until\ they$$

$$meet\ at\ point\ (\ C\ ).$$

$$The\ area\ of\ triangle\ \triangle\ ABC\ is\ a\ function\ of\ x\ ,\ denoted\ as\ \triangle\ (x).$$

$$If\ we\ extend\ the\ domain\ of\ \ \triangle\ (x)\ from\ \left[-\frac{1}{2}, 1\right]\ to\ (-\infty, \infty), evaluate\ the\ integral{:}$$

$$\int_{-\infty}^{\infty} \left(\Delta(x) + \frac{\sqrt{\ln 2}}{4}\right) dx$$

$$(Instruction{:}\ Do\ not\ use\ modulus\ for\ distances.)$$

Ans = _______________



+91-9257282136

Info@vedantsamanvaya.com

**Q11.** The condition part is subjective. It is only for tiebreakers.

Now, cut the ellipsoid with a plane perpendicular to the $x - axis$ at some fixed $x$ obtaining an ellipse $E$ in the . Next, take this ellipse $E$ and translate it parallel to the $xz - plane$ from the positive $z - axis$ to the negative $z - axis$ in such a way that a second, identical ellipse is completely enclosed within it. Find the area of the resulting shaded region formed due to this transformation. Consider a surface given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad where\ b > c > a$$

Two planar sections are made:

1. The first section is taken at $x = k$, yielding a planar curve in the $yz - plane$.
2. The second section is taken at $y = h$, yielding a planar curve in the $xz - plane$.

Let the areas of the two cross-sections be $A_1$ and $A_2$, respectively. Compute the absolute difference $|A_1 - A_2|$ in terms of $a, b, c, k, h$, and determine under what conditions one section encloses a strictly larger area than the other.

a. $\pi c \cdot \left| b\sqrt{1 - \frac{k^2}{a^2}} - a\sqrt{1 - \frac{h^2}{b^2}} \right|$

b. $\pi c \cdot \left| b\sqrt{1 - \frac{h^2}{b^2}} - a\sqrt{1 - \frac{k^2}{a^2}} \right|$

c. $\pi b \cdot \left| c\sqrt{1 - \frac{k^2}{a^2}} - a\sqrt{1 - \frac{h^2}{b^2}} \right|$



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**Q12.** *Consider a* $n-$ *dimensional Guassian manifold given by:*

$$M_n: z = e^{-(x_1^2+x_2^2+\cdots+x_n^2)}$$

Find the volume enclosed between $M_n$ and the plane $z = 0$ over the domain $x_1, x_2, \ldots, x_n \in R^n$.

For the beginners, manifold simply means surface ;)

a. $\frac{\pi^{(n-1)/2}}{2}$

b. $\frac{\pi^n}{2}$

c. $\frac{1}{2^n}\pi^{n/2}$

d. $\frac{\pi^{n/2}}{2}$

**Q13.** *Evalute the Integral*

$$I = \int_{-1}^{1} e^{-x^2}\, dx$$

*by expressing it as the limit of a sum:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=-n}^{n} e^{-\left(\frac{r}{n}\right)^2}$$

***Jensen's Inequality Statement***:

For a convex function $f$ and set a weights $(\lambda_i)$ summing to 1,

$$f\left(\sum \lambda_i x_i\right) \le \sum \lambda_i f(x_i).$$

Using Jensen's inequality with $f(x) = \ln x$, determine the best possible lower bound for $I$. Provide a numerical answer for this bound.

Ans = ____________



+91-9257282136

Info@vedantsamanvaya.com

**Q14**. In a thermodynamic system, heat $Q$ is a function of entropy $S$ and temperature $T$, given by $Q(S,T)$. It is known that entropy and temperature are independent, i.e.,

$$\frac{dS}{dT} = 0.$$

Additionally, the system satisfies the differential relations:

$$\frac{dQ}{dT} = T^5, and \ \ dS = \frac{dQ}{T}.$$

We also define another given condition:

$$\frac{dQ}{dS} = Te^T.$$

Initially, at temperature $T = 1\,K$, the heat is $Q = 0$.Find the total heat $Q$ at infinite temperature $T \rightarrow \infty$.

Additional Information for Calculation:

It is given that the infinite series

$$\sum_{r=1}^{\infty} \frac{1}{r^6} = \frac{\pi^6}{945} \approx 1.017343.$$

Use this information to compute $Q$.

Guidelines for this questions as this question represents hypothetical thermodynamic situation and may yield strange expressions.

- Start writing with $q = q(S,T)$
- Consider small change $q$ as $dq$
- Then use other given diffretial relations (Don't use diffretial relations before)
- Answer till 2 places

Ans = ____________



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**Q15.** Solve the Integral

$$\int_0^{\infty}\left(\frac{1}{\int_0^{\infty} e^{-2025\ x^{\lfloor y\rfloor}}\,dx}\right)dy$$

Where $\lfloor y \rfloor$ denotes the **floor function**, i.e., the greatest integer less than or equal to $y$.

Note exp(x) = $e^x$.

a. 5050.exp(5050)

b. 2025exp(2025)

c. 2025exp(2025)+2025

d. exp(2025)



+91-9257282136

Info@vedantsamanvaya.com

**Answers**

1. $0.616 + \frac{a}{2}$

2. 0.785398

3. $\frac{\sqrt{\pi} \cdot e^{-\left(\frac{4ac-b^2}{4a}\right)}}{\sqrt{a}}$

4. $-\frac{\pi^2}{6n}$

5. 120a

6. $1 - e^{\sin(1)}$

7. 1

8. A <= B or  A < B

9. A <= B

10. 0.53, 0.54, 0.55  any one of them acceptable

11. $\pi c \cdot \left| b\sqrt{1 - \frac{k^2}{a^2}} - a\sqrt{1 - \frac{h^2}{b^2}} \right|$

12. $\frac{\pi^{n/2}}{2}$

13. 0.51, 0.52, 0.53 any one of them acceptable

14. -122.07, -122.08, -122.09 any one of them acceptable

15. 2025exp(2025)



+91-9257282136

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