Integral Cup 2025 – Phase 1 Preliminary Exam

(PART 2)

Question Paper & Answer Key

Q1. Evaluate the integral $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^3 dx$

a. $\frac{\pi}{2}$

b. $\frac{3\pi}{8}$

c. $\frac{\pi^2}{6}$

d. $\frac{1}{2}$

Q2. Kinda obvious but c tends to 0 when x tends to 0 and all things are well defined

$$Let f(x) and g(x) be continuously differentiable functions on [0,1],$$

$$with\ g(x) > 0\ for\ all\ x \in [0,1], and\ define$$

$$F(x) = \frac{\int_0^x f(t)g(t)\,dt}{\int_0^x g(t)\,dt}$$

By the First Mean Value Theorem for Integrals, for each $x \in (0,1]$, there exists $\boldsymbol{c} \in (0,x)$ such that $F(x) = f(c)$. Suppose that f is strictly increasing and $f'(x) \neq 0$ on $(0,1)$. Then which of the following gives an explicit expression for c when x=1?

a. $\int_0^1 \frac{g(x)[f(x)\int_0^x g(t)dt - \int_0^x f(t)g(t)dt]}{(\int_0^x g(t)dt)^2 f\prime(c(x))} dx$

b. $\int_0^1 \frac{g(x)[f(x)\int_0^x g(t)dt - \int_0^x f(t)g(t)dt]}{(\int_0^1 g(t)dt)^2 f\prime(c(x))} dx$

c. $\int_0^1 \frac{g(x)[f(x)\int_0^1 g(t)dt - \int_0^1 f(t)g(t)dt]}{(\int_0^x g(t)dt)^2 f\prime(c(x))} dx$

d. $\int_0^1 \frac{g(x)[f(x)\int_0^x g(t)dt - \int_0^1 f(t)g(t)dt]}{(\int_0^x g(t)dt)^2 f\prime(c(x))} dx$

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Q3. Let f (x) be a real-valued function that is infinitely differentiable on R, and suppose its Maclaurin series converges to f(x) for all $x \in R$. Evaluate the integral:

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx$$

a. $\sqrt{\pi}\sum_{k=0}^{\infty}\frac{f^{(2k)}(0)}{k!\cdot 2^k}$

b. $\sqrt{\pi}\sum_{k=0}^{\infty}\frac{f^{(2k)}(0)}{(2k)!}$

c. $\sqrt{\pi}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!\cdot 2^k}$

d. $\sum_{k=0}^{\infty}\frac{f^{(2k)}(0)}{k!\cdot 2^k}$

Q4 . Given that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Evaluate the definite integral

$$\int_0^1 \frac{1}{x} ln\left\{\left(\frac{1+x}{1-x}\right)^2\right\}\, dx$$

a. π^2

b. $\frac{\pi^2}{2}$

c. $\frac{\pi^2}{4}$

d. $\frac{\pi^2}{6}$

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Q5. Answer in Integer

$$\lim_{x\to 0}\frac{\int_0^x \sin(f(t)^n)\,dt}{\int_0^x \sin^n(f(t))\,dt}$$

Answer- ________________

Q6. Given that $a = \int_0^1 \frac{\tan^{-1}x}{x}dx \text{ and } \zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

evaluate the integral in terms of a and π:

$$\int_0^1 \frac{\ln x}{1 - x^4}dx$$

a. $-\left(\frac{a}{2}+\frac{\pi^2}{16}\right)$

b. $\frac{a}{2}-\frac{\pi^2}{16}$

c. $-\left(a+\frac{\pi^2}{8}\right)$

d. $\frac{\pi^2}{8} - a$

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Q7. The Gamma function is defined for $\alpha > 0$ as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

Another important identity involving the Gamma function is the

reflection formula (symmetry relation):

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{sin(\pi x)}, \quad for \;\; 0 < x < 1$$

Now, consider the integral:

$$I_{\text{n}} = \int_0^\infty \frac{e^{x^{1/n}} - 1}{x(1+x)} dx$$

Evaluate the limit:

$$\lim_{n\to\infty} \frac{I_n}{n}$$

a. $\sum_{k=1}^{\infty} \frac{1}{k \cdot k!}$

b. $\sum_{k=1}^{\infty} \frac{1}{(k!)^2}$

c. $\sum_{k=1}^{\infty} \frac{1}{k^2 \cdot k!}$

d. $\sum_{k=1}^{\infty} \frac{1}{(k+1) \cdot k!}$

Q8. Evaluate the integral:

$$\int_0^1 \sqrt{-\ln(x)}\; dx$$

This is a standard integral known to have a closed-form expression involving $\sqrt{\pi}$

Use the approximation: $\sqrt{\pi} \approx 1.772$

What is the value of the integral correct to 2 decimal places ?

Answer = _____________

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Q9. Let $f(x)$ be a real-valued function defined for all $x \geq 1$, satisfying:

- $f(1) = 1$
- $f'(x) = \frac{1}{x^2+f(x)^2}$

Then the **tightest correct upper bound** on $\lim_{x\to\infty} f(x)$?

Round your answer to two decimal places.

Answer = ____________

Q10. Round your answer to two decimal places.

$$\text{Let} \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n} x^{2n} = \ln\left(\frac{\pi x}{\sin(\pi x)}\right), \quad \text{for } 0 < |x| < 1.$$

$$\text{Evaluate: } \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n+1} \cdot \left(\frac{1}{4}\right)^n$$

Answer = _______________

Q11. Let f : [0, 1] → R be a continuous, non-negative function satisfying

$$f(t)^2 \leq 1 + 2\int_0^t f(s)\, ds \quad \text{for all } t \in [0,1].$$

Which of the following is the **tightest upper bound** that can be guaranteed for f(1)?

a. $\sqrt{3}$

b. 2

c. $1 + \frac{\pi}{4}$

d. $\frac{5}{2}$

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Q12. Determine which one is greater

Let $f(x)$ be a function such that $f(0) = 0$ and $f'(0) > 0$. Consider the intervals:

$$[0, a], [a, 2a], [2a, 3a], [3a, 4a], [4a, 5a], [5a, 6a].$$

Assume that $f(ka) = 0$ for $k \ in \ \{0, 1, 2, 3, 4, 5, 6\}$. By Rolle's Theorem, there exists exactly one critical point in each interval, denoted as $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$, where:

$$f'(\alpha_k) = 0 \quad \text{for } k \in \{1,2,3,4,5,6\}.$$

Furthermore, it is given that:

$$\left|\int_0^a f(x)\,dx + \int_{2a}^{3a} f(x)\,dx + \int_{4a}^{5a} f(x)\,dx\right| < \left|\int_a^{2a} f(x)\,dx + \int_{3a}^{4a} f(x)\,dx + \int_{5a}^{6a} f(x)\,dx\right|.$$

Which of the following expressions is necessarily positive?

a. $\big(f(\alpha_1) + f(\alpha_2)\big) \cdot \big(f(\alpha_3) + f(\alpha_4)\big) \cdot \big(f(\alpha_5) + f(\alpha_6)\big)$

b. $\big(f(\alpha_1) + f(\alpha_2)\big) \cdot \big(f(\alpha_3) + f(\alpha_4)\big) \cdot \big(f(\alpha_5) + f(\alpha_6)\big)$

c. $|f(\alpha_2) + f(\alpha_4) + f(\alpha_6)| - \big(f(\alpha_1) - f(\alpha_3) - f(\alpha_5)\big)$

d. $f(\alpha_1)f(\alpha_2) + f(\alpha_3)f(\alpha_4) + f(\alpha_5)f(\alpha_6)$

Q13. Compare the following two integrals and Determine which one is greater:

$$A = \int_{-\infty}^{\infty} e^{x(i-x)}\,dx$$

$$B = \int_{-\infty}^{\infty} e^{-x^2} \cos x\,dx$$

a) A<B

b) A>B

c) A=B

d) Can't say

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Q14. Let f and g be real-valued functions defined and twice continuously differentiable on $[0,1]$, such that:

$$f(0) = g(0) = 0$$

Let c $\in (0,1)$ be the point guaranteed by the **Cauchy's Mean Value Theorem (CMVT)**, which states that for such functions,

$$\exists\, c \in (0,1) \text{ such that } \frac{f'(c)}{g'(c)} = \frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f(1)}{g(1)}$$

Further, suppose:

$$\int_0^1 \frac{7f'(x) - 8g'(x)}{2g(x) - 5f(x)}\, dx = c$$

Then find the value of the sum:

$$f'(c) + g'(c) + f''(c) + g''(c)$$

Note:- You may assume all expressions are well-defined and all denominators are non-zero on the interval.

Answer = ____________________

Q15. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin(ax)\sin(bx)}{x^2}\, dx$$

a. $\pi \max(a, b)$

b. $\frac{\pi}{2}(a + b)$

c. $\pi \min(a, b)$

d. $\frac{2\pi ab}{a+b}$

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Answers

1. $\frac{3\pi}{8}$

2. $\int_0^1 \frac{g(x)[f(x)\int_0^x g(t)dt - \int_0^x f(t)g(t)dt]}{(\int_0^x g(t)dt)^2 f\prime(c(x))} dx$

3. $\sqrt{\pi}\sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{k! \cdot 2^k}$

4. $\frac{\pi^2}{2}$

5. 1

6. $-\left(\frac{a}{2} + \frac{\pi^2}{16}\right)$

7. $\sum_{k=1}^{\infty} \frac{1}{k \cdot k!}$

8. 0.88 or 0.89

9. 1.78 or 1.79

10. 0.15 or 0.16

11. 2

12. Opt-c

13. A=B

14.20

15.$\pi \min(a, b)$

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